

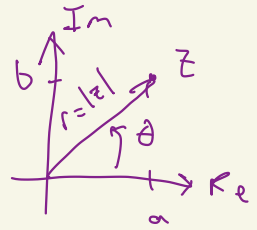
Wednesday, October 1, Lecture 14: Series

Last time... Complex numbers

$$z = a + bi \quad (\text{rectangular})$$

$$= r e^{i\theta} \quad (\text{polar})$$

$$(\cos\theta + i\sin\theta)$$



Multiplication by z :

- Rescales by $r = |z|$
- Rotates by θ (in $e^{i\theta}$)

Ex: Solns to $z^2 = 2i$

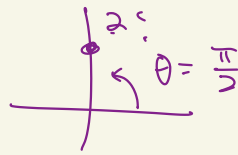
Last time (rectangular): two solns,



Let's try polar.

$$(re^{i\theta})^2 = 2e^{i\frac{\pi}{2}}$$

$$r^2 e^{i2\theta}$$



. Take norm of both sides

$$|r^2 e^{i2\theta}| = |2e^{i\frac{\pi}{2}}|$$

$$r^2 = 2$$

$$\Rightarrow r = \sqrt{2}, \text{ one soln (since } r > 0)$$

. Take arg of both sides:

$$\arg(r^2 e^{i2\theta}) = \arg(2e^{i\frac{\pi}{2}})$$

$$2\theta \bmod 2\pi = \frac{\pi}{2} \bmod 2\pi$$

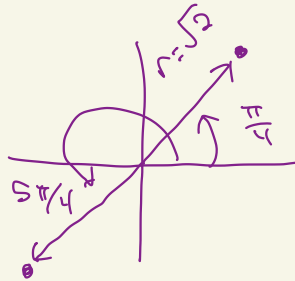
$$\Rightarrow \theta = \frac{\pi}{4} \text{ or } \theta = \frac{5\pi}{4}$$

$$(\text{since } 2 \cdot \frac{\pi}{4} = \frac{\pi}{2})$$

$$\text{and } 2 \cdot \frac{5\pi}{4} = \frac{10\pi}{4}$$

$$= \frac{\pi}{2} + 2\pi$$

$$= \frac{\pi}{2} \bmod 2\pi$$



We will later discuss the fundamental theorem of algebra, which guarantees every degree n polynomial has exactly n roots, counting multiplicity.

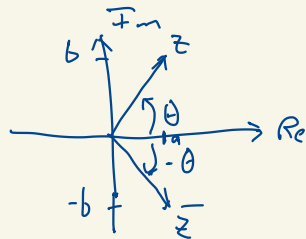
Complex conjugation

Def:

The complex conjugate $\bar{z}$ of $z \in \mathbb{C}$ is

$$\bar{z} = \overline{a+ib} = a-ib$$

$$\bar{z} = \overline{re^{i\theta}} = re^{-i\theta}$$



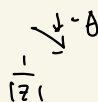
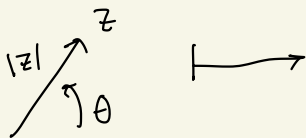
Useful fact:

$$|z|^2 = z\bar{z}$$

Pf:

$$\begin{aligned} (a+ib)\overline{(a+ib)} &= (a+ib)(a-ib) = a^2 + iab - iba - i^2b \\ &= a^2 + b^2 \\ &= |a+ib|^2. \quad \square \end{aligned}$$

$$\Rightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2} \leftarrow \text{angle} = -\theta, \text{ and } \left| \frac{\bar{z}}{|z|^2} \right| = \frac{|\bar{z}|}{|z|^2} = \frac{|z|}{|z|^2} = \frac{1}{|z|}$$



Power series

Just like real functions, we can define power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{what does converge mean?}$$

Def:

- We say a sequence of complex numbers z_n converges to $z \in \mathbb{C}$

if $\lim_{n \rightarrow \infty} |z - z_n| = 0$. (so $\lim_{n \rightarrow \infty} z_n = z$).

real numbers!

- Equivalently, $z_n \rightarrow z$ if $z_n = a_n + ib_n$ and $z = a + ib$ where

$$\lim_{n \rightarrow \infty} a_n = a \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = b,$$

Def:

- we say a power series $\sum a_n z^n$ converges

the sequence of partial sums $S_N(z) = \sum_{n=0}^N a_n z^n$ converges.

- $\sum a_n z^n$ converges absolutely if $\sum |a_n z^n|$ converges.

- Automatically means exponential and geometric series converge on their radii of convergence.

exp: $\frac{|z|^n}{n!} \quad \forall z \in \mathbb{C}$

$\nwarrow$ real number

$\sin, \cos \dots$

geom: $|z| < 1$
 $R=1$



- All of the old tests apply! Just use triangle inequality a lot:

$$|z+w| \leq |z| + |w| \quad \left| \sum a_n z^n \right| \leq \sum |a_n| |z|^n$$

Observation:

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots \\ &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \\ &\quad + iz - i \frac{z^3}{3!} + i \frac{z^5}{5!} - \dots \\ &= \cos z + i \sin z \end{aligned}$$

Thm (Euler's Formula)

For any $z \in \mathbb{C}$,

$$e^{iz} = \cos z + i \sin z.$$

Special cases: if $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{Rotation.}$$

$$\begin{aligned} |e^{i\theta}| &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= 1 \end{aligned}$$

Notice:

$$e^{iz} = \cos z + i \sin z$$

$$e^{-iz} = \cos(-z) + i \sin(-z) = \cos z - i \sin z$$

$$\Rightarrow \begin{aligned} \bullet \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \bullet \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Warning: be careful when extending from $\mathbb{R}$ to $\mathbb{C}$:

$$\sin x \rightarrow \sin z$$

$$|\sin x| \leq 1 \text{ for all } x \in \mathbb{R}$$

but

$$|\sin i2\pi| = \left| \frac{e^{i(i2\pi)} - e^{-i(i2\pi)}}{2i} \right| = \frac{1}{2} |e^{-2\pi} - e^{2\pi}|$$

$$\approx 268.$$

Not too hard to see that $\sin z, \cos z$ unbounded in imaginary direction $i\mathbb{R}$ (just like e on $\mathbb{R}$).

This also supplies us with a definition:

when $z = iy \in i\mathbb{R}$ (z is purely imaginary),

$$\sin iy = \frac{e^{-y} - e^y}{2i} =: i \sinh(y)$$

$$\cos iy = \frac{e^{-y} + e^y}{2} =: \cosh(y)$$

"Hyperbolic
Trig"

• Indeed, these share many properties with $\sin, \cos$ because they are $\sin$ and $\cos$... restricted to the imaginary axis.